F. anal. 10. 2. 2025

- info / slides(?)

A. Ergodicity, irrational no's and Weyl

(Appl. of F.-theory.)

Transformation on $[0,1)$:

$$\rho : [0,1) \longrightarrow [0,1) \; ; \; \rho^n = \underbrace{\rho \circ \cdots \circ \rho}_{n\text{-times}}$$

(Discr.) dyn. syst. on $[0,1]$:

(1) $\quad x_n = \rho^n(x_0), \quad n = 0, 1, \ldots$

(1) ergodic if:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{\infty} f(\rho^n(x)) = \int_0^1 f(y)\,dy \quad \forall x \in [0,1], \; f \in C([0,1])$$

time-average = space average

Important concept in statistical physics!

Simple model:

$$\rho : x \longmapsto x + \gamma \mod 1 = \langle x+\gamma \rangle, \quad \gamma \in (0,1)$$

and

(2) $\quad x_n = \langle x_0 + n\gamma \rangle, \quad n = 0, 1, \ldots$

Rem. 1:

$$x = \underset{\substack{\text{integer}\\ \text{part}}}{[x]} + \underset{\substack{\text{frac.}\\ \text{part}}}{\langle x \rangle} \quad , \quad [x] = \max\{a \in \mathbb{N} : a \leq x\}$$

$$\langle x \rangle = x - [x] \in [0,1)$$

Ex.: $\pi = [\pi] + \langle \pi \rangle = 3 + 0{,}1415\ldots$

$-\pi = [-\pi] + \langle -\pi \rangle = -4 + 0{,}9394\ldots$

$$x \bmod 1 := \langle x \rangle \in [0,1) \quad .$$

(mod 1 def. equiv. rel'n - see book)

Lem. 1 (Weyl): $\underset{\text{irrational}}{\gamma \in (0,1) \setminus \mathbb{Q}} \Rightarrow (2) \in$ ergodic

Obs: f 1-periodic $\Rightarrow f(\langle x \rangle) = f([x] + \langle x \rangle) = f(x)$

$$\Rightarrow f(\rho^n(x)) = f(\langle x + n\gamma \rangle) = f(x + n\gamma)$$

Hence Lem. 1 equivalent to:

Lem. 2: $\gamma \in (0,1) \setminus \mathbb{Q}$; $x \in \mathbb{R}$; $f \in C(\mathbb{R})$, 1-per.

$$(3) \quad \Rightarrow \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x + n\gamma) = \int_0^1 f(y)\, dy \, ,$$

Rem. 2. p. 3

Pf.:

1) (3) holds for $f = e_m$, $e_m = e^{2\pi i m x}$: (basis for $L^2(0,1)$)

$m = 0$: $\frac{1}{N} \sum_{n=1}^{N} 1 = 1 = \int_0^1 1$ ok

$m \neq 0$: $e_m(\gamma) = e^{2\pi i m \gamma} \neq 1$

Rem. 2 :

(c) Lem. 2 not true for $\gamma = \frac{p}{q}$; $p, q \in \mathbb{N}$:

$$\tilde{f}(x) = \sin^2(2\pi q x)$$

Chk. : $\tilde{f}(x+n\gamma) = \tilde{f}(x)$, $\int_0^1 \tilde{f}(y)\,dy \neq 0$

$$\frac{1}{N}\sum_{n=1}^{N} \tilde{f}(x+n\gamma) = \tilde{f}(x) \neq 0 \text{ for most } x \in \mathbb{R}$$

(a) $$\sum_{n=1}^{N} f(x+n\gamma)\frac{1}{N} \overset{N>>1}{=} \sum_{n=1}^{N} f(x_n)\,h \simeq \int_0^1 f(y)\,dy$$

when $\{x_n = \langle x+n\gamma\rangle\}_{n=1}^{\infty} \simeq \{nh\}_{n=1}^{\infty}$, $h = \frac{1}{N}$

So $\{x_n\}_{n=1}^{N}$ should be "equi-distributed" on $[0,1]$ and dense as $N \to \infty$

(b) $\gamma \in \mathbb{Q} \cap (0,1) \Rightarrow \gamma = \frac{p}{q}$; $p, q \in \mathbb{N}$

Then $x_n = x_{n+q}$, q-per. , and <u>not</u> equi-distr. / dense as $N \to \infty$.

$$\frac{1}{N}\sum_{n=1}^{N} e_m(x+n\gamma) = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m(x+n\gamma)}$$

$$= \frac{1}{N}\underbrace{e^{2\pi i m(x+\gamma)}\frac{1-e^{2\pi i m N\gamma}}{1-e^{2\pi i m\gamma}}}_{\text{bdd in } N}$$

$$\xrightarrow[N\to\infty]{} 0 \underset{\text{chk.}}{=} \int_0^1 e_m(x)\,dx$$

2) (3) holds for trig. poly. $P(x) = \sum_{|j|\le k} c_j e^{i2\pi jx}$:

$$\frac{1}{N}\sum_{n=1}^{N} P(x+n\gamma) = \sum_{|j|\le k}\left(\frac{1}{N}\sum_{n=1}^{N} e_j(x+n\gamma)\right) \xrightarrow[N\to\infty]{1)} c_0\cdot 1 + \sum_{0<|j|\le k} c_j\cdot 0$$

$$\overset{\text{chk.}}{=} \int_0^1 P(y)\,dy$$

3) (3) holds for $f\in C(\mathbb{R})$:

Approx: $\exists P_k$ trig. poly. s.t. $\|f-P_k\|_\infty \xrightarrow[k\to\infty]{} 0$

(e.g. $P_k = \sigma_k(f)$, Cesarò sum)

$$e_N := \left|\frac{1}{N}\sum_{n=1}^{N} f(x+n\gamma) - \int_0^1 f(y)\,dy\right|$$

$$\le \underbrace{\left|\frac{1}{N}\sum_{n=1}^{N} f(x+n\gamma) - \frac{1}{N}\sum_{n=1}^{N} P_k(x+n\gamma)\right|}_{\le \|f-P_k\|_\infty \to 0,\ k\to\infty \text{ (indep. of } N)}$$

$$+ \underbrace{\left|\frac{1}{N}\sum_{n=1}^{N} P_k(x+n\gamma) - \int_0^1 P_k(y)\,dy\right|}_{=: I_k(N) \xrightarrow[N\to\infty]{} 0 \text{ (k fixed)}}$$

$$+ \underbrace{\left|\int_0^1 P_k(y)\,dy - \int_0^1 f(y)\,dy\right|}_{\le \|P_k - f\|_\infty \xrightarrow[k\to\infty]{} 0}$$

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$\forall \varepsilon > 0$ take first k large $\|f - P_k\|_\infty < \frac{\varepsilon}{3}$,

then N large s.t. $I_k(N) < \frac{\varepsilon}{3}$, thence

$$e_N < \|f - P_k\|_\infty + I_k(N) + \|P_k - f\|_\infty < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \square$$

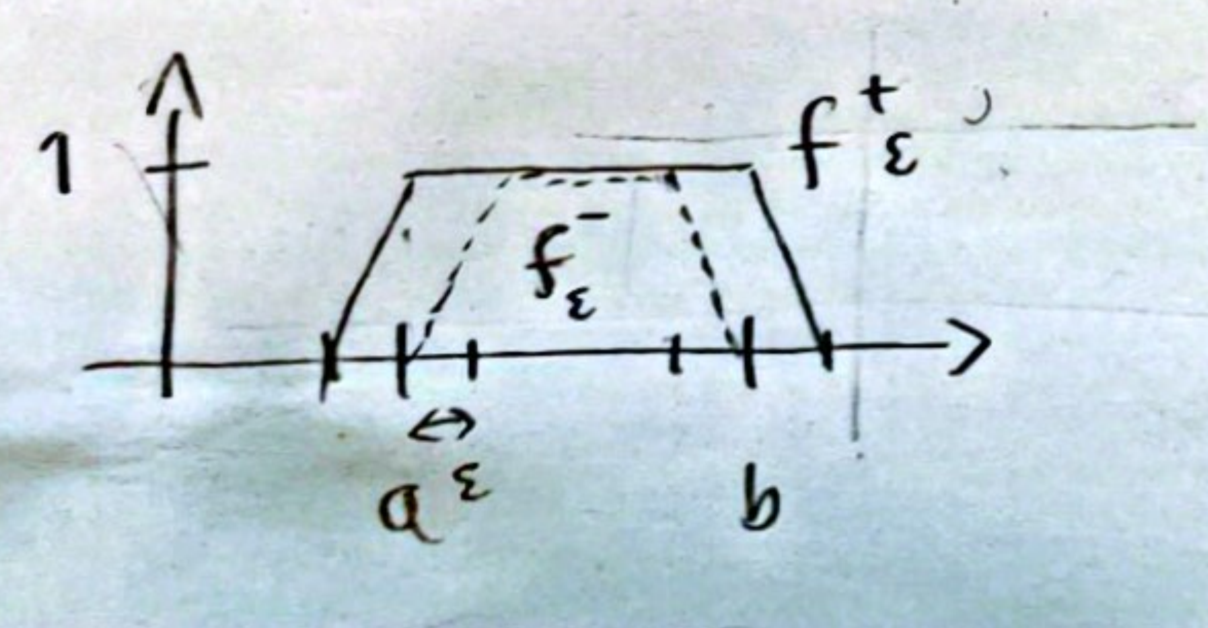
By approx., Lem. 2 holds for

$$f(x) = \chi_{(a,b)}(x), \qquad (a,b) \subset (0,1)$$

where $\chi_{(a,b)}$ is 1-per. extension of $\chi_{(a,b)} = \begin{cases} 1, & x \in (a,b) \\ 0, & x \in [0,1] \setminus (a,b) \end{cases}$

Let $\varepsilon > 0$ and $f_\varepsilon^\pm \in C(\mathbb{R})$, 1-per., p.w. lin. s.t.

$$\chi_{(a+\varepsilon, b-\varepsilon)} \le f_\varepsilon^- \le \chi_{(a,b)} \le f_\varepsilon^+ \le \chi_{(a-\varepsilon, b+\varepsilon)}$$



Obs:

$$\frac{1}{N}\sum_{n=1}^{N} f_\varepsilon^-(x+n\gamma) \le \underbrace{\frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(x+n\gamma)}_{S_N} \le \frac{1}{N}\sum_{n=1}^{N} f_\varepsilon^+(x+n\gamma)$$

Lem. 2 $\downarrow N \to \infty$ $\qquad\qquad$ $\downarrow N \to \infty$

$$\int_0^1 f_\varepsilon^- \qquad\qquad \int_0^1 f_\varepsilon^+$$

$$\text{VI} \qquad\qquad \wedge\!\text{I}$$

$$b - a - 2\varepsilon \qquad\qquad b - a + 2\varepsilon$$

$$\Longrightarrow \; b - a - 2\varepsilon \le \liminf_{N\to\infty} S_N \le \limsup_{N\to\infty} S_N \le b - a + 2\varepsilon$$

Since $\varepsilon > 0$ can be arbitrary small, we have:

Cor. 1: $\gamma \in (0,1) \setminus \mathbb{Q}$; $(a,b) \subset (0,1)$, $x \in [0,1)$

$$\Rightarrow \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(x+n\gamma) \xrightarrow[N \to \infty]{} \int_0^1 \chi_{(a,b)} = b-a$$

Rem. 3:

Cor. 1 $\Rightarrow$ Lem. 2 holds for R. int. f'ns

[since step-f'ns approx. R.-int. f'ns both pt. wise and in L^1 (see SS)]

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B. Weyl's equidistribution theorem

~~$x = \underbrace{[x]}_{\text{integer part}} + \underbrace{\langle x \rangle}_{\text{frac. part}}$, $\langle x \rangle \in [0,1)$~~

Let $x_0 \in [0,1)$, $\gamma \in (0,1)$, and

$$x_n = \langle x_0 + n\gamma \rangle, \quad n = 0, 1, \ldots$$

Obs 1:

(a) $\gamma \in (0,1) \cap \mathbb{Q}$, then $\{x_n\}_{n=1}^{\infty} \overset{\gamma = \frac{p}{q}}{=} \{x_0 + \gamma, \ldots, x_0 + q\gamma\}$ finite set

(b) $\gamma \in (0,1) \setminus \mathbb{Q}$, then all x_n distinct!

[if not: $\langle x_0 + n_1\gamma \rangle = \langle x_0 + n_2 \gamma \rangle \Rightarrow n_1\gamma - n_2\gamma \in \mathbb{Z}$

$\Rightarrow \gamma \in \mathbb{Q}$, contradiction]

It turns out $\{x_n\}_n \subset (0,1)$ is dense and equi-distr.

Def. 1: $\{\xi_n\}_{n=1}^{\infty} \subset [0,1)$ equidistributed if

$$\forall\, (a,b) \subset [0,1), \quad \lim_{N \to \infty} \underbrace{\frac{\#\{1 \le n \le N : \xi_n \in (a,b)\}}{N}}_{\substack{\text{frequency of times } \xi_n \in (a,b) \\ \text{for } n \le N}} = \underbrace{b-a}_{\frac{|(a,b)|}{|[0,1)|}}$$

where $\# A$ = cardinality of A = number of elements in A.

As $N \to \infty$, every interval in $[0,1)$ of same length, is visited by ξ_n equally often.

Ex. 1:

$$0, \tfrac{1}{2}, 0, \tfrac{1}{3}, \tfrac{2}{3}, 0, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, 0, \tfrac{1}{5}, \tfrac{2}{5}, \dots$$

is equi-distr.

Ex. 2: $\{r_n\}_n = \mathbb{Q} \cap [0,1)$, $\xi_n = \begin{cases} r_{n/2}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

$\{\xi_n\}_{n=1}^{\infty}$ dense in $[0,1)$, but not equi-distr.

(half of seq'nce is at 0)

Thm. 1: Weyl equi-distr. thm

$\gamma \in \mathbb{R} \setminus \mathbb{Q}$, $x \in [0,1)$

$$\Rightarrow \{\langle x + n\gamma \rangle\}_{n=1}^{\infty} \text{ equi-distr.}$$

Stop here

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A. Weyl's equidistribution theorem

$\{\xi_n\}_{n=1}^{\infty} \subset [0,1)$ equi-distr. if $\forall\, (a,b) \subset [0,1)$,

$$\rho_\xi((a,b)) := \lim_{N\to\infty} \underbrace{\frac{\#\{1 \le n \le N : \xi_n \in (a,b)\}}{N}}_{=\rho_\xi(N) \,=\, \text{frequency of times } \xi_n \in (a,b) \text{ for } n \le N} = \underbrace{b-a}_{\frac{|(b,a)|}{|[0,1)|}}$$

Recall: $\underset{\mathbb{R}}{\overset{\cap}{x}} = \underset{\mathbb{Z}}{\overset{\cap}{[x]}} + \underset{[0,1)}{\overset{\cap}{\langle x\rangle}}$, integer + frac. part.

Thm. 1: Weyl's equi-distr. thm.

$\gamma = (0,1) \setminus \mathbb{Q}$, $x \in [0,1)$ $\Rightarrow$ $\{\langle x + n\gamma\rangle\}_{n=1}^{\infty}$ equi-distr.

Rem. 1:

(a) $\gamma \in (0,1) \cap \mathbb{Q}$ $\Rightarrow$ $\{\langle x+n\gamma\rangle\}_{n=1}^{\infty}$ finite, not equi-distr.

(b) ξ_n equi-distr. $\Rightarrow$ ξ_n visits (a,b) w. frequency $b-a \in (0,1)$ as $n \to \infty$.

$N = 5$ ... $N \gg 1$

(c) $a < b \Rightarrow \rho_\xi((a,b)) = b - a > 0$

$\Rightarrow \xi_n \in (a,b)$ for ∞ many n's

(d) equi-distr. $\Rightarrow$ dense.
$\not\Leftarrow$

$\Rightarrow$) Let $\bar{x} \in [0,1)$. $I_n := (\bar{x} - \frac{1}{n}, \bar{x} + \frac{1}{n})$

$\rho_\xi(I_n) = \frac{2}{n} > 0 \Rightarrow \exists\, m_n$ s.t. $\xi_{m_n} \in I_n$

Obs: $\xi_{m_n} \xrightarrow[n\to\infty]{} \bar{x}$. Hence $\{\xi_m\}_n$ dense. since $\bar{x}$ a.b.

$\not\Leftarrow$) $\xi_n = \begin{cases} r_{\frac{n}{2}} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ $\{r_n\}_n = \mathbb{Q} \cap (0,1)$

dense, but $\rho_\xi((\frac{1}{n}, 1)) < \frac{1}{2}$ $\forall n \in \mathbb{N}$.

<u>Cor. 1:</u> Kronecker's thm.

$\gamma \in (0,1) \setminus \mathbb{Q}$, $x \in [0,1)$ $\Rightarrow \{\langle x + n\gamma \rangle\}_{n=1}^{\infty}$ dense in $[0,1)$

<u>Pf:</u> Thm. 1 + Rem 1 (d) $\square$

Pf. Thm. I:

Obs: $\#\{1 \le n \le N : \langle x_0 + n\gamma \rangle \in (a,b)\}$

$$= \sum_{n=1}^{N} \chi_{(a,b)}(x_0 + n\gamma)$$

1-per. extension of $\chi_{(a,b)} = \begin{cases} 1, & x \in (a,b) \\ 0, & x \in [0,1) \setminus (a,b) \end{cases}$

Cor. 1 last time

$$\frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(x_0 + n\gamma) \xrightarrow[N \to \infty]{} \int_0^1 \chi_{(a,b)} = b - a \qquad \square$$

Weyl's criterion:

$$\{\xi_n\}_{n=1}^{\infty} \subset [0,1) \text{ equidistr.} \iff \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} \xrightarrow[N \to \infty]{} 0 \quad \forall k \in \mathbb{Z},\ k \neq 0$$

Pf.:

$\Leftarrow$) $\frac{1}{N} \sum e^{2\pi i k \xi_n} \to 0$ replaces step 1

in pf. of Lem 2 last time.

The rest of the pf is then the same.

$\square$

$\Rightarrow$) Exercise 5

Ex. I: (Exercise 6 - use Weyl's crit.)

$\{\langle a n^{\sigma} \rangle\}_{n=1}^{\infty}$ equidistr. when $a \neq 0$, $\sigma \in (0,1)$

$\{\langle a \log n \rangle\}_{n=1}^{\infty}$ not equidistr. for any a

B. Intermezzo: dominated conv. for series and termwise differentiation

Thm. 2: Assume $f_{n,m}, f_n \in C([a,b])$ satisfy for all $n \in \mathbb{Z}$:

(i) $\|f_{m,n} - f_n\|_\infty \xrightarrow[m\to\infty]{} 0$, (ii) $\sup_m \|f_{m,n}\|_\infty \le M_n$, $\sum_{n\in\mathbb{Z}} M_n < \infty$

Then $\sum_{n\in\mathbb{Z}} f_n(x)$ conv. unif. and abs., $\sum_{n\in\mathbb{Z}} f_n(x) \in C([a,b])$, and

$$\Big\|\sum_{n\in\mathbb{Z}} f_{m,n} - \sum_{m\in\mathbb{Z}} f_n\Big\|_\infty \xrightarrow[m\to\infty]{} 0.$$

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So: $\lim_m \sum f_{m,n} = \sum \lim_m f_{m,n}$ "both sums conv. $\sum|f_{m,n}| \le \sum M_n$, $\sum|f_n| \le 2\sum M_n$"

Pf.:

$$\Big\|\sum_{n\in\mathbb{Z}} f_{m,n} - \sum_{n\in\mathbb{Z}} f_n\Big\|_\infty \le \underbrace{\sum_{|n|\le N}\|f_{m,n}-f_n\|_\infty}_{<\frac{\varepsilon}{2}\text{ for } m \text{ large and } N \text{ fixed by (i)}} + \underbrace{\sum_{|n|>N}\Big(\underbrace{\|f_{m,n}\|_\infty}_{\le M_n} + \underbrace{\|f_n\|_\infty}_{\le \underbrace{\|f_n - f_{m_n,n}\|_\infty}_{<\frac{\varepsilon}{2}\ \le M_n \text{ for } m_n \text{ large}} + \underbrace{\|f_{m_n,n}\|_\infty}_{\le M_n}}\Big)}_{\le 3\sum_{|n|>N} M_n < \frac{\varepsilon}{2} \text{ for } N \text{ large by (ii)}}$$

$\square$

Cor. 2: $f_n \in C^1([a,b])$, $\|f_n'\| \le M_n$, $\sum_{n\in\mathbb{Z}} M_n < \infty$

$$\Rightarrow \frac{d}{dx}\sum_{n\in\mathbb{Z}} f_n(x) \text{ exists, is cont., and } = \sum_{n\in\mathbb{Z}} f_n'(x) \text{ (conv. abs + unif.)}$$

Pf.: $f_{m,n}(x) := \dfrac{f(x+\frac{1}{m}) - f(x)}{\frac{1}{m}} = \int_0^1 f_n'(x + t\tfrac{1}{m})\,dt$

$\|f_{m,n}\|_\infty \le \int_0^1 \|f_n'\|_\infty\,dt \le M_n$

$$\|f_{m,n} - f_n'\|_\infty \le \int_0^1 \underbrace{\sup_{x} |f_n'(x+t\tfrac{1}{m}) - f_n'(x)|}_{\le \omega_{f_n'}(t\cdot\frac{1}{m})}\,dt \le \omega_{f_n'}\big(\tfrac{1}{m}\big) \xrightarrow[m\to\infty]{} 0 \qquad t\in(0,1)$$

Conclude by applying Thm. 2 $\square$

(Problem: $x = b \Rightarrow x + t\frac{1}{m} \notin [a,b]$. Sol'n: extend f to $f^e \in C^1([a,b+1])$ s.t. $f = f^e$ on $[a,b]$)

Alt.: Redef. $f_{m,n} = m\begin{cases} f(x+\frac{1}{m}) - f(x), & x < \frac{1}{2} \\ f(x) - f(x-\frac{1}{m}), & x > \frac{1}{2}\end{cases}$

C. Heat eq'n on $(0,1)$ (sirkel)

$$\begin{cases} u_t = u_{xx}, \quad t \in (0,T), \quad x \in (0,1) & (1) \\ u(0,t) = u(1,t) \quad \text{per. B.C.} \quad (u \text{ f'n on circle}) & (2) \\ u(x,0) = f(x) \in R^1 \ (\text{or } L^1) \quad \text{I.C.} & (3) \end{cases}$$

Separation of var's $\xrightarrow[\text{Compl. f'n theory}]{\text{Math 4}}$ candidate sol'n:

$$(4) \quad v(x,t) := \sum_{n \in \mathbb{Z}} \underbrace{\underbrace{\hat{f}(n)\, e^{-4\pi^2 n^2 t}}_{\hat{v}(n,t)} e^{2\pi i n x}}_{v_n(x,t)}$$

Rapidly decaying F. coeff.'s and v_n's:

$$(5) \quad |v_n(x,t)| \leq |\hat{v}(n,t)| \leq \underbrace{\sup |\hat{f}(n)|}_{\substack{\leq \|f\|_1 \ (\leq \|f\|_2 \leq \|f\|_\infty) \\ < \infty}} \underbrace{\left(n^{k+2} e^{-4\pi^2 n^2 t}\right)}_{\substack{\leq \sup_n (\dots) \\ = C_{k,t} < \infty \text{ for } t > 0}} \frac{1}{n^{k+2}} \quad \forall k \in \mathbb{N}$$

Obs. I:

(a) $v_n(x,t) \in C^\infty([0,1] \times (0,T))$

$$\partial_t^k \partial_x^m v_n = (-4\pi^2 n^2)^k (2\pi i n)^m v_n \in C^\infty \quad \forall k, m \in \mathbb{N}$$

$$|\text{— " —}(x,t)| \underbrace{\overset{(5)}{\leq}}_{\leq |n|^{2k+m} |v_n(x,t)|} \|f\|_1 \, C_{2k+m,\varepsilon} \frac{1}{n^2} =: M_n \quad \text{for } t \in (\varepsilon, T),\ x \in [0,1]$$

(b) $\partial_x^2 v_n(x,t) = (2\pi i n)^2 v_n = -4\pi^2 n^2 v_n = \partial_t v_n(x,t) \quad \forall n$

$v_n(0,t) = v_n(1,t)$

Lem. 1: $v \in C^\infty([0,1] \times (0,T))$ and satisfy (1) and (2).

Pf.:

Repeated use of Cor. 2 and Obs. 1 (a) gives

$$\partial_t^k \partial_x^m \sum v_n = \sum \partial_t^k \partial_x^m v_m \text{ is cont. for } t > \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary $v \in C^\infty([0,1] \times (0,T))$.

$$v_{xx} \overset{\text{Cor. 2}}{=} \sum v_{n,xx} \overset{\text{Obs. 1 (b)}}{=} \sum v_{n,t} \overset{\text{Cor. 2}}{=} v_t \quad , \; t > 0, x \in (0,1)$$

Chk: $\sum v_n(0,t) = \sum v_n(1,t)$ by Obs. 1 (b). $\square$

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abs. conv. of series. $\square$

What about I.C. ?

Obs. 2: $v(x,t) = (f * H_t)(x)$ where H_t heat kernel

on circle $(0,1)$.

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

[solves (1) - (3) w. $f = \delta(x)$]

Later: H_t good kernel, as $t \to 0$, and hence

$$v(x,t) = (f * H_t)(x) \xrightarrow[t \to 0]{} f(x) \text{ at } x \text{ where } f \text{ cont.}$$